

Answer any one from the followings: $10 \times 1 = 10$
Briefly describe the sustainable use of resources with suitable example.

Analyse the limits to growth theory with criticism.

714/Math.

UG/6th Sem/MATH-H-DSE-T-03B/23

U.G. 6th Semester Examination - 2023

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-03B

(Number Theory)

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

The symbols and notations have their usual meanings.

1. Answer any ten questions: $2 \times 10 = 20$
- If n is an odd integer, show that $n^4 + 4n^2 + 11$ is of the form $16k$.
 - Verify that if an integer is simultaneously a square and a cube, then it must be either of the form $7k$ or $7k + 1$.
 - Prove that the product of four consecutive integers is 1 less than a perfect square.
 - Show that for a positive integer n and any integer a , $\gcd(a, a + n)$ divides n .

- e) If $\gcd(a, b) = 1$, prove that $\gcd(a^2, b^2) = 1$.
- f) Using Euclidean Algorithm, obtain integers x and y such that $\gcd(1769, 2378) = 1769x + 2378y$.
- g) Find all prime numbers that divide $100!$.
- h) Prove that the only prime of the form $n^2 - 4$ is 5.
- i) Prove that if $n > 2$, then there exists a prime p satisfying $n < p < n!$.
- j) For any integer a , show that $a^2 - a + 7$ ends in one of the digits 3, 7 or 9.
- k) Prove that $53^{103} + 103^{53}$ is divisible by 39.
- l) Solve $x^2 + 7x + 10 = 0 \pmod{11}$.
- m) For what values of n does $n!$ exist?
- n) Find a prime number p that is simultaneously expressible in the forms $x^2 + y^2$, $u^2 + 2v^2$, and $r^2 + 3s^2$.
- o) Show that there are no positive integers n , such that $\sigma(n) = 10$.
2. Answer any four questions: $5 \times 4 = 20$
- a) Given integers a and b , not both of which are zero, prove that there exist integers x and y such that $\gcd(a, b) = ax + by$.
- b) Assuming that p_n is the n th prime number, show that the sum $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n}$ is never an integer.
- c) Show that, the quadratic congruence $x^2 + 1 \equiv 0 \pmod{p}$, where p is an odd prime, has a solution if and only if $p \equiv 1 \pmod{4}$.
- d) Prove that if p is prime, then $(p-1)! \equiv -1 \pmod{p}$.
- e) Find all values of $n \geq 1$ for which $n! + (n+1)! + (n+2)!$ is perfect square.
- f) State and prove the quadratic reciprocity law.
3. Answer any two questions: $10 \times 2 = 20$
- a) i) Show that if p is a prime divisor of $839 = 38^2 - 5 \cdot 11^2$, then $(5/p) = 1$. Use this fact to conclude that 839 is a prime number. 4
- ii) For an odd prime, prove that the congruence $2x^3 + 1 \equiv 0 \pmod{p}$ has a solution if and only if $p \equiv 1$ or $3 \pmod{8}$. 6
- b) i) Show that for any positive integer n , $\sigma(n)$ is an odd integer if and only if n is a perfect square or twice a perfect square. 4

- ii) Using Wilson's theorem, prove that for any odd prime p ,

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-1)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.$$

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- c) i) Establish that for any positive integer n ,

$$\frac{\sigma(n!)}{n} \geq 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

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- ii) Prove that for any positive integer r , the product of any r consecutive positive integers is divisible by $r!$

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Full Marks : 40

The figures in the right-hand margin
Candidates are required to give
their own words as far as

1. Answer any five from the following
- Mention two distinct characteristics of India.
 - Name two Hot spots of India.
 - Enumerate two adaptive strategies.
 - What is eutrophication?
 - Define Red Tide. Name one organism.
 - Name two edaphic factors in agriculture.
 - Give the full form of ICAR.